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A new approach to the momentum expansion of multiloop Feynman diagrams

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Abstract

We present a new method for the momentum expansion of Feynman integrals with arbitrary masses and any number of loops and external momenta. By using the parametric representation we derive a generating function for the coefficients of the small momentum expansion of an arbitrary diagram. The method is applicable for the expansion w.r.t. all or a subset of external momenta. The coefficients of the expansion are obtained by applying a differential operator to a given integral with shifted value of the space-time dimension d and the expansion momenta set equal to zero. Integrals with changed d are evaluated by using the generalized recurrence relations proposed in [1]. We show how the method works for one- and two-loop integrals. It is also illustrated that our method is simpler and more efficient than others.

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1. Introduction

Accurate theoretical predictions for precision experiments in particle physics are usually the result of formidable calculations of radiative corrections. More and more particles with different masses have to be taken into account. This requires the evaluation of complicated Feynman integrals with many masses and external momenta. Under lucky circumstances one can get analytic results for the radiative corrections but more frequently one is forced to expand w.r.t. external momenta and/or masses or to find other kinds of approximate calculations [2], [3]. Frequently even if the result for a particular diagram is known analytically [4], the numerical evaluation proceeds via approximations which again may be some kind of momentum expansion. It is evident that methods of asymptotic expansions of Feynman integrals [5] will play an increasing role in the evaluation of physical amplitudes.

The asymptotic momentum expansions combined with conformal mapping and subsequent resummation by means of Padè approximations were used to calculate the diagrams in the whole cut plane in the external momentum squared [6]-[9]. As was found in these papers the method gives very precise results in a wide range of kinematical variables. To extend further this method we need to find a systematic and effective method for the momentum expansion of diagrams depending on several external momenta and masses. For the three-point Green functions this problem was solved in [8], [10]. As for diagrams with more external legs no reasonable algorithm is known up to now.

The solution of this problem is important not only for the case of small momenta. Asymptotic expansions at large momenta or masses also include small momentum expansion as an ingredient part [11].

In the present paper we describe a new approach to the momentum expansion which solves the aforementioned problem. The method is closely related to the new, generalized recurrence relations for Feynman integrals proposed in [1].

The paper is organized as follows. In Sect.2, we present the main ideas of our method. First, using the parametric representation, we give a derivation of the generating function for the coefficients of the expansion of an arbitrary propagator type integral with different masses. Then we derive a similar formula for the diagrams with arbitrary number of external legs.

In Sect.3 we demonstrate how the method works for one-loop integrals. A general formula, in terms of a one-fold integral, for the coefficients of the expansion of scalar n -point one-loop diagrams is derived.

In Sect.4 several examples of the small momentum expansion of various two-loop diagrams are considered. We show how to expand integrals w.r.t. subsets of external momenta. The characteristic functions required for expanding propagator type and three-point vertex diagrams are given.

2. General formalism

The subject of our consideration will be dimensionally regulated scalar Feynman integrals. An arbitrary scalar L loop integral can be written as

$$G^{(d)}(\{s_i\}, \{m_s^2\}) = \prod_{i=1}^L \int d^d k_i \prod_{j=1}^N P_{\vec{k}_j, m_j}^{\nu_j}, \quad (1)$$

where

$$P_{k,m}^\nu = \frac{1}{(k^2 - m^2 + i\epsilon)^\nu}, \quad \vec{k}_j^\mu = \sum_{n=1}^L \omega_{jn} k_n^\mu + \sum_{m=1}^E \eta_{jm} q_m^\mu, \quad (2)$$

q_m are external momenta, $\{s_i\}$ is a set of scalar invariants formed from q_m , N is the number of lines, E is the number of external legs, ω and η are matrices of incidences of the graph with the matrix elements being ± 1 or 0 (see, for example, Ref. [12]).

In what follows we will need a parametric representation for $G^{(d)}(\{s_i\}, \{m_s^2\})$. For an arbitrary scalar Feynman integral in d dimensional space-time we have [12], [13]:

$$G^{(d)}(\{s_i\}, \{m_s^2\}) = i^L \left(\frac{\pi}{i}\right)^{\frac{dL}{2}} \prod_{j=1}^N \frac{i^{-\nu_j}}{\Gamma(\nu_j)} \int_0^\infty \cdots \int_0^\infty \frac{d\alpha_j \alpha_j^{\nu_j-1}}{[D(\alpha)]^{\frac{d}{2}}} e^{i[\frac{Q(\{s_i\}, \alpha)}{D(\alpha)} - \sum_{l=1}^N \alpha_l (m_l^2 - i\epsilon)]}, \quad (3)$$

where $D(\alpha)$ and $Q(\{s_i\}, \alpha)$ are homogeneous polynomials in α of degree L and $L+1$, respectively. They can be represented as sums over trees and two-trees of the graph (see, for example, Ref. [14]):

$$D(\alpha) = \sum_{\substack{\text{over} \\ \text{trees}}} \left(\prod_{\substack{\text{over} \\ \text{chords}}} \dots \alpha_j \dots \right),$$

$$Q(\{s_i\}, \alpha) = \sum_{\substack{\text{over} \\ 2\text{-trees}}} \prod_{\substack{\text{over} \\ \text{chords}}} \dots \alpha_j \dots \left(\sum_{\substack{\text{over comp.} \\ \text{of 2-tree}}} q \right)^2. \quad (4)$$

These polynomials are characteristic functions of the topology of the diagram and of its subgraphs. Since D and Q will play an important role in the rest of the present paper we remind the reader of the definitions of the trees and two-trees for connected diagrams. Any connected subdiagram of a diagram G containing all the vertices of G but is free of cycles (loops) is called a tree of G . Similarly, a two-tree is defined as any subdiagram of G containing all the vertices of the original diagram, but is free of cycles, and consisting of exactly two connected components. Finally, a chord of a tree (two-tree) is defined as any line not belonging to this tree (two-tree).

For the practical application of our method the following properties of the functions D and Q are important. Namely, the dependence of D and Q on each α_ν , can be written in the form:

$$D(\alpha) = D_{\overline{\nu}}(\alpha) \alpha_\nu + D_\nu(\alpha),$$

$$Q(\{s\}, \alpha) = Q_{\overline{\nu}}(\{s\}, \alpha) \alpha_\nu + Q_\nu(\{s\}, \alpha), \quad (5)$$

where $D_{\overline{\nu}}(\alpha)$, $Q_{\overline{\nu}}(\{s\}, \alpha)$ are the D , Q functions of the diagram obtained by removing the line ν , and $D_{\nu}(\alpha)$, $Q_{\nu}(\{s\}, \alpha)$ are the D , Q functions for the diagram obtained by contracting the line ν to a point. D_{ν} , $D_{\overline{\nu}}$, Q_{ν} and $Q_{\overline{\nu}}$ do not depend on α_{ν} . The polynomials $D(\alpha)$, $Q(\{s_i\}, \alpha)$ can be easily constructed by means of a computer program for any particular integral [15]. Examples of D and Q for some particular diagrams will be given in the next sections.

Let us first consider propagator type integrals. In this case Q is proportional to the external momentum squared q^2 : $Q(q^2, \alpha) = \overline{Q}(\alpha)q^2$. Assuming temporarily that all propagators have different masses, from the representation (3) we obtain:

$$\frac{\partial}{\partial q^2} G^{(d)}(q^2, \{m_s^2\}) = \frac{(-1)^{L+1}}{\pi^L} \overline{Q}(\partial) G^{(d+2)}(q^2, \{m_s^2\}), \quad (6)$$

where $\overline{Q}(\partial)$, is the differential operator obtained from $\overline{Q}(\alpha)$ by replacing $\alpha_j \rightarrow \partial_j \equiv \partial/\partial m_j^2$. In the above formula it is assumed that after differentiation all masses are set to their original value.

Thus, differentiation w.r.t. momenta of the d dimensional integral is replaced by differentiation w.r.t. masses of the $d+2$ dimensional one. As we shall show in the next sections this replacement in many cases will essentially simplify practical calculations. For the l -th derivative we can write

$$\left(\frac{\partial}{\partial q^2} \right)^l G^{(d)}(q^2, \{m_s^2\}) = \frac{(-1)^{(L+1)l}}{\pi^{Ll}} \overline{Q}^l(\partial) G^{(d+2l)}(q^2, \{m_s^2\}). \quad (7)$$

Upon setting $q^2 = 0$ we obtain the coefficients of the small momentum expansion of the propagator type diagram.

The general prescription for the small momentum expansion is then the following. First, one should assign to all lines different masses and then set the external momenta equal to zero. Second, the differential operator $Q(\partial)$ must be applied as much as needed and then the masses must be set to the original one's. Third, the integration by parts recurrence relations in $d+2l$ dimensions should be used, reducing all the integrals to the set of master integrals. Fourth, the set of all master integrals in $d+2l$ dimensions must be expressed in terms of d dimensional master integrals using the generalized recurrence relations described in [1]. In some cases it is advantageous to use recurrence relations which simultaneously reduce the value of d and the sum of exponents of the propagators in the integral. Rather frequently master integrals are just combinations of Euler's Γ functions, and transforming the latter from $d+2l$ to d dimensions is quite simple.

In practical calculations no real differentiation is needed. It is enough to compute $Q^l(\alpha)$, multiply it by $G^{(d+2l)}(0, \{m_s^2\})$ and make the substitutions

$$\frac{\alpha_i^j}{[k_s^2 - m_i^2 + i\epsilon]^n} = \frac{\Gamma(n+j)}{\Gamma(n)[k_s^2 - m_i^2 + i\epsilon]^{n+j}}.$$

An important feature of our method is that one does not need to care about external momenta, avoiding many complications intrinsic to other methods based on the differentiation w.r.t. external momenta. From the very beginning, one is dealing with

simpler objects; only bubble diagrams need to be considered. In all cases where the bubble diagrams are known, one immediately gets an explicit generating function for the coefficients.

Unfortunately the polynomial Q usually has many terms, and it is not so easy to exponentiate it. For the three-loop nonplanar master integral occurring in the QED photon propagator, Q is a fourth order polynomial with 45 terms. In this case using FORM [16] running on a DEC 3000 it took about 46 hours to generate the 12-th coefficient in the small momentum expansion in terms of bubble integrals. Nevertheless, the number of generated terms, and as a consequence the execution time, is smaller than what one gets using the direct small momentum expansion methods. Using the method proposed recently in [17] for solving recurrence relations for three-loop vacuum integrals one may expect an improvement of the efficiency in the complete calculations of the coefficients. Our method is definitely the most optimal for the case of all different masses. In case when some masses in the integral are equal the calculation can be simplified by taking into account this information from the very beginning.

The algorithm described above can be generalized to the small momentum expansion of integrals with an arbitrary number of external momenta. For integrals depending on several external momenta the polynomial Q is a sum over a set of $n = E(E-1)/2$ independent scalar invariants s_j :

$$Q(\alpha) = \sum_{j=1}^n s_j Q_j(\alpha), \quad (8)$$

where E - is the number of external legs of the diagram corresponding to a given integral. In this case the analogue of (7), valid for propagators, reads:

$$\frac{\partial^l}{\partial s_j^l} G^{(d)}(\{s_k\}, \{m_i^2\}) = \frac{(-1)^{(L+1)l}}{\pi^{Ll}} Q_j^l(\partial) G^{(d+2l)}(\{s_k\}, \{m_i^2\}). \quad (9)$$

To find the coefficient of the M -th order in $s_1^{l_1} \dots s_n^{l_n}$ one should apply the product of operators $Q_1^{l_1}(\partial) \dots Q_n^{l_n}(\partial)$ to the integral taken in $d+2M$ dimensions. If we perform an expansion w.r.t. all scalar invariants again the coefficients of the expansion will be generated from the bubble integrals. In this case the general formula for the small momentum expansion of an arbitrary scalar integral will be:

$$G^{(d)}(\{s_k\}, \{m_i^2\}) = \sum_{M=0}^{\infty} \frac{(-1)^{M(L+1)}}{\pi^{ML}} \times \sum_{\{l_1+\dots+l_n=M\}} Q_1^{l_1}(\partial) \dots Q_n^{l_n}(\partial) G^{(d+2M)}(\{0\}, \{m_i^2\}) \frac{s_1^{l_1}}{l_1!} \dots \frac{s_n^{l_n}}{l_n!}. \quad (10)$$

The expansion can also be performed w.r.t. some subset of scalar invariants which are proportional to the momenta considered to be small. In this case the operators $Q_j(\partial)$ must be applied to the given integral where only that subset of the momenta is taken to be zero. The corresponding formula is almost identical to (10) and may be readily written as:

$$G^{(d)}(\{s_k\}, \{S_F\}, \{m_i^2\}) = \sum_{M=0}^{\infty} \frac{(-1)^{M(L+1)}}{\pi^{ML}} \times \sum_{\{l_1+\dots+l_n=M\}} Q_1^{l_1}(\partial) \dots Q_n^{l_n}(\partial) G^{(d+2M)}(\{0\}, \{S_F\}, \{m_i^2\}) \frac{s_1^{l_1}}{l_1!} \dots \frac{s_n^{l_n}}{l_n!}, \quad (11)$$

where $\{S_F\}$ is a set of scalar invariants assumed to be fixed.

Equations (10) and (11) can be used not only for the small momentum expansion. The momentum expansion considered is also an essential ingredient of other kind of asymptotic expansions [9], [11], [18].

3. Expansion of one-loop integrals

Now we shall show how the method works by considering several simple examples. We start with the one-loop integral:

$$I_{\alpha\beta}^{(d)}(q^2, m_1^2, m_2^2) = \int \frac{d^d k_1}{[i\pi^{d/2}]} P_{k_1, m_1}^{\alpha} P_{k_1 - q, m_2}^{\beta} = \sum_{l=0}^{\infty} \frac{I_l}{l!} (q^2)^l. \quad (12)$$

In this case $Q(q^2, \alpha) = \alpha_1 \alpha_2 q^2$. The l -th coefficient of the small momentum expansion will be

$$I_l = \frac{\partial^l}{\partial(m_1^2)^l} \frac{\partial^l}{\partial(m_2^2)^l} \int \frac{d^{d+2l} k_1}{[i\pi^{d/2+l}]} P_{k_1, m_1}^{\alpha} P_{k_1, m_2}^{\beta}. \quad (13)$$

Upon differentiation we obtain:

$$I_l = \frac{\Gamma(l+\alpha) \Gamma(l+\beta)}{\Gamma(\alpha) \Gamma(\beta)} \int \frac{d^{d+2l} k_1}{[i\pi^{d/2+l}]} P_{k_1, m_1}^{\alpha+l} P_{k_1, m_2}^{\beta+l}. \quad (14)$$

The bubble integral can be expressed in terms of hypergeometric function ${}_2F_1$:

$$\int \frac{d^d k_1}{[\pi^{d/2}]} P_{k_1, m_1}^{\alpha} P_{k_1, m_2}^{\beta} = \frac{(-1)^{\alpha+\beta} \Gamma(\alpha + \beta - \frac{d}{2})}{(m_1^2)^{\alpha - \frac{d}{2}} (m_2^2)^{\beta} \Gamma(\alpha + \beta)} {}_2F_1 \left[\begin{matrix} \beta, \frac{d}{2}; \\ \alpha + \beta; \end{matrix} \middle| 1 - \frac{m_1^2}{m_2^2} \right]. \quad (15)$$

For simplicity we will take $m_1^2 = m_2^2 = m^2$. In this case

$$I_l = (-1)^{\alpha+\beta} \frac{\Gamma(l+\alpha) \Gamma(l+\beta)}{\Gamma(\alpha) \Gamma(\beta)} \frac{\Gamma(\alpha + \beta + l - \frac{d}{2})}{\Gamma(\alpha + \beta + 2l) (m^2)^{l+\alpha+\beta-d/2}}. \quad (16)$$

and therefore

$$I_{\alpha\beta}^{(d)}(q^2, m^2, m^2) = \frac{(-1)^{\alpha+\beta} \Gamma(\alpha + \beta - \frac{d}{2})}{(m^2)^{\alpha+\beta-\frac{d}{2}} \Gamma(\alpha + \beta)} {}_3F_2 \left[\begin{matrix} \alpha, \beta, \alpha + \beta - \frac{d}{2}; \\ \frac{\alpha+\beta}{2}, \frac{\alpha+\beta+1}{2}; \end{matrix} \middle| \frac{q^2}{4m^2} \right], \quad (17)$$

in agreement with Ref.[19].

To compare our method with [20] it is instructive to calculate the coefficient I_1 at $m_1^2 = m_2^2 = m^2$, $\alpha = \beta = 1$ using both techniques. In our approach I_1 is just one term:

$$\begin{aligned} I_1 &= \frac{\partial^2}{\partial m_1^2 \partial m_2^2} \int \frac{d^{d+2}k_1}{[i\pi^{d/2+1}]} \frac{1}{(k_1^2 - m_1^2 + i\epsilon)(k_1^2 - m_2^2 + i\epsilon)} \Big|_{m_1^2=m_2^2=m^2} \\ &= \int \frac{d^{d+2}k_1}{[i\pi^{d/2+1}]} \frac{1}{(k_1^2 - m^2 + i\epsilon)^4}. \end{aligned} \quad (18)$$

The same coefficient in the method [20] is given by

$$\begin{aligned} I_1 &= \frac{1}{2d} \frac{\partial^2}{\partial q_\mu \partial q^\mu} \int \frac{d^d k_1}{[i\pi^{d/2}]} \frac{1}{((k_1 - q)^2 - m^2 + i\epsilon)(k_1^2 - m^2 + i\epsilon)} \Big|_{q=0} \\ &= \frac{1}{d} \int \frac{d^d k_1}{[i\pi^{d/2}]} \frac{[4k_1^2 - d(k_1^2 - m^2)]}{(k_1^2 - m^2 + i\epsilon)^4}. \end{aligned} \quad (19)$$

One can observe at first, that differentiating w.r.t. masses is easier than w.r.t. external momenta. Secondly, after differentiation the number of terms for I_1 in (19) is larger than in our approach. The number of terms for the higher coefficients I_l in the approach [20] grows exponentially. A very similar situation we find for integrals with more loops.

For d dimensional one-loop integrals corresponding to diagrams with E legs the number of terms in Q is equal to the number of independent scalar invariants $n = E(E-1)/2$. In this case Q may be represented as:

$$Q(\{s\}, \alpha) = \sum_{j=1}^n Q_j(\alpha) s_j, \quad (20)$$

with $Q_j(\alpha)$ being monomials in α . In the small momentum expansion of the one-loop integrals:

$$\begin{aligned} G^{(d)}(\{s_i\}, \{m_s^2\}) &= \int d^d k_1 P_{k_1 - q_1, m_1}^{\nu_1} \cdots P_{k_1 - q_{E-1}, m_{E-1}}^{\nu_{E-1}} P_{k_1, m_E}^{\nu_E} \\ &= \sum_{M=0}^{\infty} \sum_{\{l_1 + \dots + l_n = M\}} C_{l_1 \dots l_n} \frac{s_1^{l_1}}{l_1!} \cdots \frac{s_n^{l_n}}{l_n!}, \end{aligned} \quad (21)$$

the coefficients of the expansion will be

$$C_{l_1 \dots l_n} = \frac{1}{\pi^M} Q_1^{l_1}(\partial) \cdots Q_n^{l_n}(\partial) \int d^{d+2M} k_1 P_{k_1, m_1}^{\nu_1} \cdots P_{k_1, m_E}^{\nu_E}. \quad (22)$$

Performing the angular integration we obtain a compact one-fold integral representation for the coefficients of the momentum expansion of an arbitrary one-loop integral:

$$C_{l_1 \dots l_n} = \frac{i\pi^{\frac{d}{2}-M} Q_1^{l_1}(\partial) \cdots Q_n^{l_n}(\partial)}{\Gamma(\frac{d}{2} + M)} \int_0^\infty \frac{(-1)^{\nu_1 + \dots + \nu_E} x^{\frac{d}{2} + M - 1} dx}{(x + m_1^2)^{\nu_1} \cdots (x + m_E^2)^{\nu_E}}. \quad (23)$$

The previous integral may readily be evaluated in terms of the Lauricella function $F_D^{(E-1)}$ (see formula 7.1.1.5 in Ref. [21]) in agreement with Ref. [22]. In the case when

all masses are equal or some masses are zero while the others are equal the integral can be evaluated in terms of Euler's Γ function.

For illustration, we consider the expansion of the three-point vertex function w.r.t. external momenta q_1^2 , q_2^2 , $q_3^2 = (q_1 + q_2)^2$:

$$I_{\nu_1\nu_2\nu_3} = \int \frac{d^d k_1}{[i\pi^{d/2}]} P_{k_1-q_1, m_1}^{\nu_1} P_{k_1, m_2}^{\nu_2} P_{k_1+q_2, m_3}^{\nu_3} = \sum_{l_1, l_2, l_3=0}^{\infty} C_{l_1 l_2 l_3} \frac{(q_1^2)^{l_1} (q_2^2)^{l_2} (q_3^2)^{l_3}}{l_1! l_2! l_3!}. \quad (24)$$

The function Q corresponding to $I_{\nu_1\nu_2\nu_3}$ reads:

$$Q(\{s\}, \alpha) = \alpha_1 \alpha_2 q_1^2 + \alpha_2 \alpha_3 q_2^2 + \alpha_1 \alpha_3 q_3^2. \quad (25)$$

The coefficients of the expansion in q_1^2 , q_2^2 , q_3^2 may readily be found from (22) and (23):

$$\begin{aligned} C_{l_1 l_2 l_3} &= (\partial_1 \partial_2)^{l_1} (\partial_2 \partial_3)^{l_2} (\partial_1 \partial_3)^{l_3} \int \frac{d^{d+2l_1+2l_2+2l_3} k_1}{[i\pi^{d/2+l_1+l_2+l_3}]} P_{k_1, m_1}^{\nu_1} P_{k_1, m_2}^{\nu_2} P_{k_1, m_3}^{\nu_3} \\ &= \frac{(\nu_1)_{l_1+l_3} (\nu_2)_{l_1+l_2} (\nu_3)_{l_2+l_3}}{\Gamma(\frac{d}{2} + l_1 + l_2 + l_3)} \int_0^\infty \frac{(-1)^{\nu_1+\nu_2+\nu_3} x^{\frac{d}{2}+l_1+l_2+l_3-1} dx}{(x+m_1^2)^{\nu_1+l_1+l_3} (x+m_2^2)^{\nu_2+l_1+l_2} (x+m_3^2)^{\nu_3+l_2+l_3}}, \end{aligned} \quad (26)$$

where $(\nu)_l \equiv \Gamma(\nu + l)/\Gamma(\nu)$ is the Pochhammer symbol. For the case of equal masses $m_1^2 = m_2^2 = m_3^2 = m^2$, the coefficients are just products of Euler's Γ functions:

$$C_{l_1 l_2 l_3} = (-1)^{\nu_1+\nu_2+\nu_3} \frac{(\nu_1)_{l_1+l_3} (\nu_2)_{l_1+l_2} (\nu_3)_{l_2+l_3} \Gamma\left(\nu_1 + \nu_2 + \nu_3 + l_1 + l_2 + l_3 - \frac{d}{2}\right)}{(m^2)^{\nu_1+\nu_2+\nu_3+l_1+l_2+l_3-\frac{d}{2}} \Gamma(\nu_1 + \nu_2 + \nu_3 + 2(l_1 + l_2 + l_3))}. \quad (27)$$

Let us consider now an integral with one small momentum, and the other ones arbitrary. This kind of kinematics occurs, for instance, in the evaluation of the moments of the deep inelastic structure functions [23]. The one-loop scalar diagram which is of interest in this case is shown in Fig.1.

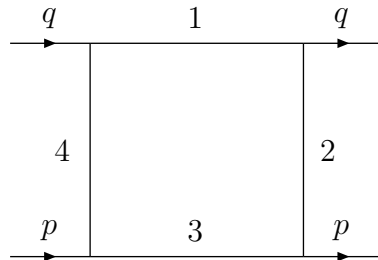


Fig.1 Box diagram depending on two momenta

The contribution to the moments of the structure functions from the integral related to the diagram in Fig.1 will be determined by the coefficients of the expansion w.r.t. the scalar product (pq) with the assumption $p^2 = 0$:

$$T(p, q) = \int \frac{d^d k_1}{[i\pi^{d/2}]} P_{k_1, m}^2 P_{k_1+q, m} P_{k_1-p, m} = \sum_{n=0}^{\infty} (pq)^n T_n. \quad (28)$$

From the function Q of the diagram we derive the differential operator that generates the coefficients T_n :

$$Q_{pq}(\partial) = 2\partial_1\partial_3. \quad (29)$$

Introducing auxiliary masses, setting $p = 0$, applying $Q_{pq}^l(\partial)$ to the diagram and setting $m_1^2 = m_2^2 = m_3^2 = m_4^2 = m^2$ after differentiation, we get:

$$\begin{aligned} T_n &= 2^n n! \int \frac{d^{d+2n} k_1}{[i\pi^{d/2+n}]} P_{k_1, m}^{n+3} P_{k_1+q, m}^{n+1} \\ &= \frac{2^n n! \Gamma\left(n + 4 - \frac{d}{2}\right)}{(m^2)^{n+4-d/2}} {}_3F_2\left[\begin{matrix} n+3, n+1, n+4 - \frac{d}{2}; \\ n+2, n+\frac{5}{2}; \end{matrix} \frac{q^2}{4m^2}\right]. \end{aligned} \quad (30)$$

The integral in (30) was evaluated with the aid of (17). A different method [24] to calculate moments of this kind of diagram, consists in applying a differential projection operator built up from the momenta p and q . From the above comparison made for the integral $I_{\alpha\beta}^{(d)}(q^2, m_1^2, m_2^2)$, the analysis given in the next section for the two-loop diagrams and the remark in Sect.2 concerning three-loop calculations, we may expect our method to be more efficient than the method of projections.

4. Expansion of two-loop integrals

Small momentum expansions of various kinds of two-loop diagrams have been considered in [6]–[9], [20]. For the generic two-loop propagator type diagram

$$J_{\nu_1\nu_2\nu_3\nu_4\nu_5}(q^2) = \iint \frac{d^d k_1 d^d k_2}{[i\pi^{d/2}]^2} P_{k_1, m_1}^{\nu_1} P_{k_2, m_2}^{\nu_2} P_{k_1-q, m_3}^{\nu_3} P_{k_2-q, m_4}^{\nu_4} P_{k_1-k_2, m_5}^{\nu_5}, \quad (31)$$

$Q(q^2, \alpha)$ and $D(\alpha)$ are, respectively, the third and the second order polynomials in α :

$$Q(q^2, \alpha) = \overline{Q}(\alpha)q^2 = [(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)\alpha_5 + \alpha_1\alpha_2(\alpha_3 + \alpha_4) + \alpha_3\alpha_4(\alpha_1 + \alpha_2)]q^2, \quad (32)$$

$$D(\alpha) = \alpha_5(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4). \quad (33)$$

If, at $q^2 = 0$, $J_{\nu_1\nu_2\nu_3\nu_4\nu_5}(q^2)$ does not have singularities then it can be expanded into the Taylor series:

$$J_{\nu_1\nu_2\nu_3\nu_4\nu_5}(q^2) = \sum_{l=0}^{\infty} J_l (q^2)^l. \quad (34)$$

The differential operator generating J_l will be a four-fold sum:

$$\overline{Q}^l(\partial) = \sum_{j=0}^l \sum_{r=0}^j \sum_{i=0}^j \sum_{s=0}^{l-i} \frac{l! j! (l-i)! \partial_1^{i+s} \partial_2^{l+s} \partial_3^{l-j} \partial_4^{r+l-j} \partial_5^{j-i}}{(l-j)! (j-i)! (j-r)! i! r! s! (l-i-s)!}. \quad (35)$$

Applying this operator to the diagram taken at $q = 0$ we find:

$$\begin{aligned} J_l &= \sum_{j=0}^l \sum_{r=0}^j \sum_{i=0}^j \sum_{s=0}^{l-i} \frac{j! (l-i)! (\nu_1)_{i+s} (\nu_2)_{l+s} (\nu_3)_{l-j} (\nu_4)_{r+l-j} (\nu_5)_{j-i}}{(l-j)! (j-i)! (j-r)! i! r! s! (l-i-s)!} \\ &\quad \times \iint \frac{d^{d+2l} k_1 d^{d+2l} k_2}{[i\pi^{d/2+l}]^2} P_{k_1, m_1}^{\nu_1+i+s} P_{k_2, m_2}^{\nu_2+l+s} P_{k_1, m_3}^{\nu_3+l-j} P_{k_2, m_4}^{\nu_4+r+l-j} P_{k_1-k_2, m_5}^{\nu_5+j-i}. \end{aligned} \quad (36)$$

If ν_i ($i = 1, \dots, 4$) are integer, performing partial fraction decomposition like

$$\begin{aligned} & \frac{1}{(k_1^2 - m_1^2 + i\epsilon)^{\nu_1} (k_1^2 - m_3^2 + i\epsilon)^{\nu_3}} \\ &= \sum_{j=0}^{\nu_1-1} \frac{(-1)^j (\nu_3 - 1 + j)!}{j! (\nu_3 - 1)!} \frac{1}{(m_1^2 - m_3^2)^{\nu_3+j} (k_1^2 - m_1^2 + i\epsilon)^{\nu_1-j}} \\ &+ \sum_{j=0}^{\nu_3-1} \frac{(-1)^{\nu_1} (\nu_1 - 1 + j)!}{j! (\nu_1 - 1)!} \frac{1}{(m_1^2 - m_3^2)^{\nu_1+j} (k_1^2 - m_3^2 + i\epsilon)^{\nu_3-j}}, \quad (37) \end{aligned}$$

we can express J_l as a multiple sum each term of which is proportional to a bubble integral with only three denominators raised to some powers. These integrals can be expressed in terms of four Appell's function F_4 [20], which in turn are double series. Representation (36) may be helpful for numerical calculations using the multiple precision program [25].

It took 5 min on a PC Pentium 90, using FORM [16], to generate in terms of bubble integrals the 30-th coefficient of the small momentum expansion for the two-loop master diagram with all masses different. Again, as we observed in the one-loop case, the number of generated terms in our approach is smaller than in the direct small momentum expansion. For instance, the 1-st, 2-nd, 3-rd, 4-th and 5-th coefficients of the Taylor expansion of the two-loop diagram with all masses equal have 3, 6, 10, 15 and 21 terms, respectively, in our approach and correspondingly 5, 18, 45, 97 and 182 terms in the other methods [8], [20]. To reduce the bubble integrals

$$J_{\nu_1 \nu_2 \nu_3}^{(d+2M)} = \iint \frac{d^{d+2M} k_1 d^{d+2M} k_2}{[i\pi^{d/2+M}]^2} P_{k_1, m_1}^{\nu_1} P_{k_1 - k_2, m_2}^{\nu_2} P_{k_2, m_3}^{\nu_3}, \quad (38)$$

encountered in the evaluation of the M -th coefficient of the expansion, to the $d = 4 - 2\epsilon$ dimensional ones, it was enough to use only one recurrence relation which is given in [1]:

$$\nu_2 \nu_3 d J_{\nu_1 \nu_2+1 \nu_3+1}^{(d+2)} + 2\nu_1 m_1^2 J_{\nu_1+1 \nu_2 \nu_3}^{(d)} - (d - 2\nu_1) J_{\nu_1 \nu_2 \nu_3}^{(d)} = 0. \quad (39)$$

Needless to say that in our method the coefficients were expressed in terms of integrals with the sum of exponents of scalar propagators larger than in the other methods, i.e. more complicated integrals. Despite of this fact, the sum of those integrals were calculated faster. For example, the 6-th coefficient of the aforementioned diagram was calculated two times faster. For higher coefficients, our method becomes even more effective comparing with methods known up to now. Significant improvement in the efficiency of the method may be achieved by implementing it as a numerical procedure proposed in [7] using the multiple precision program [25].

In some simple cases the method proposed may be helpful in finding analytic results. For example, the two-loop diagram with three denominators:

$$T^{(d)}(q^2, m_1^2, m_2^2, m_3^2) = \iint d^d k_1 d^d k_2 P_{k_1, m_1}^{\nu_1} P_{k_1 - k_2, m_2}^{\nu_2} P_{k_2 - q, m_3}^{\nu_3}, \quad (40)$$

has a rather simple Q function: $Q(q^2, \alpha) = \alpha_1 \alpha_2 \alpha_3 q^2$. The coefficients of the small momentum expansion of the diagram:

$$T^{(d)}(q^2, m_1^2, m_2^2, m_3^2) = \sum_{l=0}^{\infty} \frac{T_l}{l!} (q^2)^l \quad (41)$$

can be written as:

$$T_l = \frac{(-1)^l}{\pi^{2l}} (\nu_1)_l (\nu_2)_l (\nu_3)_l \iint d^{d+2l} k_1 d^{d+2l} k_2 P_{k_1, m_1}^{\nu_1+l} P_{k_1-k_2, m_2}^{\nu_2+l} P_{k_2, m_3}^{\nu_3+l}. \quad (42)$$

Using formula (4.3) from Ref. [20], we obtain:

$$\begin{aligned} T_l = & \frac{\pi^{d/2-2d} (-1)^l (-m_3^2)^{d-l-\nu_1-\nu_2-\nu_3}}{\Gamma(\nu_1) \Gamma(\nu_2) \Gamma(\nu_3) \Gamma(\frac{d}{2}+l)} \\ & \times \left\{ \Gamma(\frac{d}{2}-\nu_1) \Gamma(\frac{d}{2}-\nu_2) \Gamma(\nu_1+\nu_2+l-\frac{d}{2}) \Gamma(\nu_1+\nu_2+\nu_3+l-d) \right. \\ & \quad \times F_4(\nu_1+\nu_2+\nu_3+l-d, \nu_1+\nu_2+l-\frac{d}{2}; \nu_1+1-\frac{d}{2}, \nu_2+1-\frac{d}{2} \mid x, y) \\ & \quad + y^{\frac{d}{2}-\nu_2} \Gamma(\frac{d}{2}-\nu_1) \Gamma(\nu_2-\frac{d}{2}) \Gamma(\nu_1+l) \Gamma(\nu_1+\nu_3+l-\frac{d}{2}) \\ & \quad \times F_4(\nu_1+l, \nu_1+\nu_3+l-\frac{d}{2}; \nu_1+1-\frac{d}{2}, \frac{d}{2}+1-\nu_2 \mid x, y) \\ & \quad + x^{\frac{d}{2}-\nu_1} \Gamma(\nu_1-\frac{d}{2}) \Gamma(\frac{d}{2}-\nu_2) \Gamma(\nu_2+l) \Gamma(\nu_2+\nu_3+l-\frac{d}{2}) \\ & \quad \times F_4(\nu_2+l, \nu_2+\nu_3+l-\frac{d}{2}; \frac{d}{2}+1-\nu_1, \nu_2+1-\frac{d}{2} \mid x, y) \\ & \quad + x^{\frac{d}{2}-\nu_1} y^{\frac{d}{2}-\nu_2} \Gamma(\nu_1-\frac{d}{2}) \Gamma(l-\frac{d}{2}) \Gamma(\nu_3+l) \Gamma(\frac{d}{2}+l) \\ & \quad \left. \times F_4(\nu_3+l, \frac{d}{2}+l; \frac{d}{2}+1-\nu_1, \frac{d}{2}+1-\nu_2 \mid x, y) \right\}, \quad (43) \end{aligned}$$

where F_4 is Appell's hypergeometric function, $x = m_1^2/m_3^2$ and $y = m_2^2/m_3^2$. By inserting (43) into (41) and using the series representation for F_4 we reproduced the explicit result given in [4].

At the two-loop level, there are two topologically different three-point vertex diagrams shown in Figs. 2a and 2b, where each line corresponds to a scalar propagator with an arbitrary exponent.

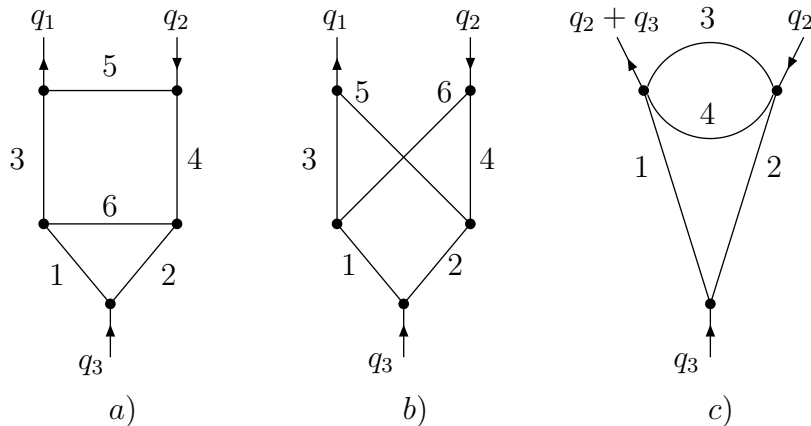


Fig.2 Two – loop vertex diagrams

The $Q(\{s\}, \alpha)$ and $D(\alpha)$ polynomials for the planar diagram Fig.2a are:

$$\begin{aligned} Q_{pl}(\{s\}, \alpha) = & (\alpha_1\alpha_3 + \alpha_1\alpha_6 + \alpha_2\alpha_3 + \alpha_3\alpha_6)\alpha_5q_1^2 \\ & + (\alpha_1\alpha_4 + \alpha_2\alpha_4 + \alpha_2\alpha_6 + \alpha_4\alpha_6)\alpha_5q_2^2 \\ & + [\alpha_1\alpha_2\alpha_5 + \alpha_6(\alpha_2 + \alpha_4)(\alpha_1 + \alpha_3) + \alpha_1\alpha_3(\alpha_2 + \alpha_4) + \alpha_2\alpha_4(\alpha_1 + \alpha_3)] q_3^2, \end{aligned} \quad (44)$$

$$D_{pl}(\alpha) = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4 + \alpha_5) + \alpha_6(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5). \quad (45)$$

For the nonplanar diagram Fig.2c $Q(\alpha)$ and $D(\alpha)$ are:

$$\begin{aligned} Q_{npl}(\{s\}, \alpha) = & [\alpha_1\alpha_5\alpha_6 + \alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_5(\alpha_1 + \alpha_2 + \alpha_4 + \alpha_6)] q_1^2 \\ & + [\alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_5\alpha_6 + \alpha_4\alpha_6(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5)] q_2^2 \\ & + [\alpha_2\alpha_3\alpha_6 + \alpha_1\alpha_4\alpha_5 + \alpha_1\alpha_2(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)] q_3^2, \end{aligned} \quad (46)$$

$$D_{npl}(\alpha) = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) + (\alpha_3 + \alpha_5)(\alpha_4 + \alpha_6). \quad (47)$$

Any two-loop scalar diagram at zero momenta $G^{(d)}(\{0\}, \{m_k^2\})$ can be written, using partial fraction decomposition (37), as a multifold sum over bubble integrals with three denominators only. As we already mentioned those integrals for arbitrary space-time dimension and arbitrary exponents of propagators are combinations of four Appell functions F_4 . Therefore, using (10), the coefficients of the Taylor expansion of any two-loop integral can be written in a closed form as a multifold sum over functions F_4 times product of Euler's Γ functions. Multiple sums will be produced by $Q_j^l(\partial)$ and partial fraction decomposition of the propagators with the same momenta and different masses.

The representation of the coefficients in terms of F_4 functions may not be of great use in general. For the cases when some masses are equal or vanish one can get simpler representations. If, for example, two masses in the vacuum integral with three denominators are equal, then the F_4 functions are reduced to the somewhat simpler ${}_4F_3$ functions [20]. If one mass is zero then F_4 reduce to ${}_2F_1$ functions.

As an example, we consider the small momentum expansion w.r.t. to the subset of scalar invariants q_3^2, q_2q_3 for the diagram Fig.2c:

$$\begin{aligned} J(q_2, q_3) &= \iint \frac{d^d k_1 d^d k_2}{[i\pi^{d/2}]^2} P_{k_1-q_2, m_1} P_{k_1-k_2, m_2} P_{k_2-q_3, m_3} P_{k_2, m_4} \\ &= \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} \frac{(2q_2q_3)^l (q_3^2)^r}{l! r!} C_{lr}(q_2^2). \end{aligned} \quad (48)$$

By using (5), the polynomial $Q(\alpha)$ for this diagram

$$Q(\{q_i q_j\}, \alpha) = 2(q_2q_3)\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2(\alpha_3 + \alpha_4)q_2^2 + q_3^2\alpha_3[\alpha_1\alpha_2 + \alpha_4(\alpha_1 + \alpha_2)], \quad (49)$$

can be obtained from (44) or (46) by setting $\alpha_3 = \alpha_4 = 0$ and then replacing $\alpha_5 \rightarrow \alpha_3$, $\alpha_6 \rightarrow \alpha_4$. The coefficients $C_{lr}(q_2^2)$ can be represented as:

$$\begin{aligned} C_{lr}(q_2^2) = & (-1)^{l+r} r! \sum_{j=0}^r \sum_{s=0}^j \frac{\partial_1^{l+r-s} \partial_2^{l+r-j+s} \partial_3^r \partial_4^j}{(r-j)!(j-s)!s!} \\ & \times \iint \frac{d^{d+2r+2l} k_1 d^{d+2r+2l} k_2}{[i\pi^{d/2+l+r}]^2} P_{k_1-q_2, m_1} P_{k_1-k_2, m_2} P_{k_2, m_3} P_{k_2, m_4}. \end{aligned} \quad (50)$$

By differentiating and then performing partial fraction decomposition, the coefficients $C_{lr}(q_2^2)$ will be expressed as combinations of integrals:

$$J_{\nu_1\nu_2\nu_3}^{(d+2r+2l)} = \iint \frac{d^{d+2r+2l}k_1 d^{d+2r+2l}k_2}{[i\pi^{d/2+l+r}]^2} P_{k_1-q_2, m_1}^{\nu_1} P_{k_1-k_2, m_2}^{\nu_2} P_{k_2, m_3}^{\nu_3}, \quad (51)$$

where m_j corresponds to m_3 or m_4 . To reduce $d + 2r + 2l$ dimensional integrals to d dimensional the generalized recurrence relations proposed in [1] must be used. Just for illustration we derive several relations of this kind. In particular one may get the relation:

$$\begin{aligned} \nu_1 q_{2\mu} J_{\mu, \nu_1+1 \nu_2\nu_3}^{(d)} + (d - \nu_1 - \nu_2 - \nu_3) J_{\nu_1\nu_2\nu_3}^{(d)} \\ - m_1^2 \nu_1 J_{\nu_1+1 \nu_2\nu_3}^{(d)} - m_2^2 \nu_2 J_{\nu_1\nu_2+1 \nu_3}^{(d)} - m_3^2 \nu_3 J_{\nu_1\nu_2\nu_3+1}^{(d)} = 0, \end{aligned} \quad (52)$$

where

$$J_{\mu, \nu_1\nu_2\nu_3}^{(d)} = \iint d^d k_1 d^d k_2 P_{k_1, m_1}^{\nu_1} P_{k_1-k_2, m_2}^{\nu_2} P_{k_2, m_3}^{\nu_3} k_{1\mu}. \quad (53)$$

Following the procedure described in [1] the integral $J_{\mu, \nu_1\nu_2\nu_3}^{(d)}$ can be expressed in terms of a scalar integral with the space-time dimension $d + 2$:

$$J_{\mu, \nu_1+1 \nu_2\nu_3}^{(d)} = q_{2\mu} \partial_2 \partial_3 J_{\nu_1+1 \nu_2\nu_3}^{(d+2)}. \quad (54)$$

By inserting (54) into (52) we obtain:

$$\begin{aligned} \nu_1 \nu_2 \nu_3 q_2^2 J_{\nu_1+1 \nu_2+1 \nu_3+1}^{(d+2)} = (\nu_1 + \nu_2 + \nu_3 - d) J_{\nu_1\nu_2\nu_3}^{(d)} \\ + m_1^2 \nu_1 J_{\nu_1+1 \nu_2\nu_3}^{(d)} + m_2^2 \nu_2 J_{\nu_1\nu_2+1 \nu_3}^{(d)} + m_3^2 \nu_3 J_{\nu_1\nu_2 \nu_3+1}^{(d)}. \end{aligned} \quad (55)$$

From the equation connecting d and $d - 2$ dimensional integrals given in [1], which in our case reads

$$J_{\nu_1\nu_2\nu_3}^{(d-2)} = (\partial_1 \partial_2 + \partial_1 \partial_3 + \partial_2 \partial_3) J_{\nu_1\nu_2\nu_3}^{(d)} \quad (56)$$

another relation follows:

$$J_{\nu_1\nu_2\nu_3}^{(d-2)} - \nu_1 \nu_2 J_{\nu_1+1 \nu_2+1 \nu_3}^{(d)} - \nu_1 \nu_3 J_{\nu_1+1 \nu_2\nu_3+1}^{(d)} - \nu_1 \nu_3 J_{\nu_1\nu_2+1 \nu_3+1}^{(d)} = 0. \quad (57)$$

By using Eqs. (55) and (57), several others obtained in a similar manner and relations derived from the method of integration by parts [26], one can reduce the integrals (51) to the set of $d = 4 - 2\varepsilon$ dimensional master integrals.

As a final example we consider the small momentum expansion of the two-loop scalar vertex diagram [6] occurring in the process $H \rightarrow 2\gamma$:

$$I = \iint \frac{d^d k_1 d^d k_2}{[i\pi^{d/2}]^2} P_{k_1+q_1, m} P_{k_1+q_2, m} P_{k_2+q_1, m} P_{k_2+q_2, m} P_{k_1-k_2, 0} P_{k_2, m}, \quad (58)$$

with $q_1^2 = q_2^2 = 0$. For this kinematics the method of integration by parts gives:

$$I = \frac{2}{(d-4)} \iint \frac{d^d k_1 d^d k_2}{[i\pi^{d/2}]^2} P_{k_1+q_1, m} P_{k_1+q_2, m}^2 P_{k_2+q_1, m} P_{k_2, m} [P_{k_2+q_2, m} - P_{k_1-k_2, 0}]. \quad (59)$$

The first term is just the product of factorized one-loop integrals which are known in terms of hypergeometric functions [19]. By using our method we expand the second integral in (59) w.r.t. $q_3^2 = (q_1 - q_2)^2$. The Q function can be obtained from (44) by setting $q_1^2 = q_2^2 = 0$ and $\alpha_4 = 0$:

$$Q(q_3^2, \alpha) = Q_3(\alpha) q_3^2 = \alpha_2 [\alpha_1(\alpha_3 + \alpha_5) + \alpha_6(\alpha_1 + \alpha_3)] q_3^2. \quad (60)$$

The differential operator for the l -th term of the expansion will be:

$$Q_3^l(\partial) = \frac{l!}{\pi^{2l}} \partial_2^l \sum_{j=0}^l \sum_{i=0}^{l-j} \sum_{k=0}^j \frac{\partial_1^{l-j+k} \partial_3^{l-k-i} \partial_5^i \partial_6^j}{(l-j-i)! (j-k)! i! k!}. \quad (61)$$

By applying this operator to the integral

$$\iint \frac{d^{d+2l} k_1 d^{d+2l} k_2}{[i\pi^{d/2}]^2} P_{k_1, m_1} P_{k_2, m_2} P_{k_1, m_3} P_{k_1, m_5} P_{k_1 - k_2, m_6},$$

setting $m_1 = m_2 = m_3 = m$, $m_6 = 0$ and adding the contribution from the factorized integrals we obtain the series expansion of I in $q_3^2 = (q_1 - q_2)^2$:

$$I = m^{12-2d} \sum_{l=0}^{\infty} I_l \left(\frac{q_3^2}{m^2} \right)^l, \quad (62)$$

where

$$I_l = \frac{2(l+1)!}{(d-4)} \left[\frac{\Gamma^2(\frac{1}{2})}{(l+1)! 4^{l+2}} \sum_{j=0}^l \frac{\Gamma(l-j+3-\frac{d}{2}) \Gamma(j+3-\frac{d}{2})}{\Gamma(l-j+\frac{3}{2}) \Gamma(j+\frac{3}{2}) (j+1)} + \frac{\Gamma(l+6-d)}{\Gamma(\frac{d}{2}+l)} \right. \\ \left. \times \sum_{j=0}^l \frac{j! \Gamma(\frac{d}{2}+l-j-1)}{\Gamma(l+j+7-d)} \sum_{k=0}^j \frac{(l-k)! \Gamma(k+3-\frac{d}{2}) \Gamma(l+j-k+4-\frac{d}{2})}{(l-j)! (j-k)! (k+1)! (2l-k+2)!} \right]. \quad (63)$$

For $d = 4$ this formula confirms the coefficients given in [6].

5. Conclusions

We presented a new systematic method for the momentum expansion of arbitrary scalar multiloop Feynman diagrams. The method can be used for the small momentum expansion and as an ingredient for the large momentum expansion or other kind of asymptotic expansions. An important feature of our method is that it can be used in cases when only some momenta are small and the others arbitrary. This extends the applicability of our method to a wider class of physical problems. Of special interest will be the application of our method to the evaluation of the moments of structure functions in deep inelastic scattering.

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